SADLER MATHEMATICS SPECIALIST UNIT 1

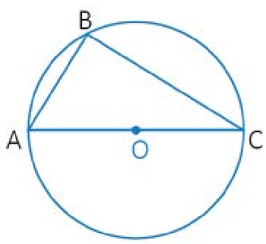
WORKED SOLUTIONS

Chapter 5 Geometric proof

Exercise 5A

Question 1

RTP: For the given diagram, $\angle ABC = 90°$



Proof:

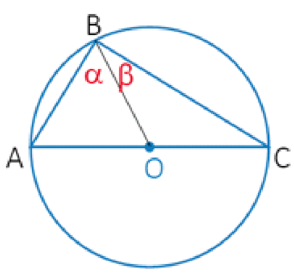
$\angle AOC = 180°$ $\angle AOC$ is a straight angle as AC is a diameter

$\angle ABC = 90°$ Angle at the centre is twice the angle at the circumference

Alternatively

RTP: For the given diagram, $\angle ABC = 90°$

Construction: OB radius



Proof:

$OA = OB = OC$ Radii of same circle

Let $\angle OBA = \alpha$, $\angle OBC = \beta$

$\angle OAB = \alpha$ ΔOAB is isosceles

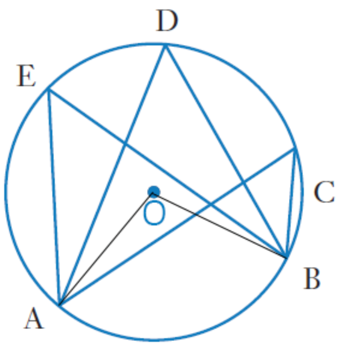
$\angle OCB = \beta$ ΔOBC is isosceles

$\angle OAB + \angle ABC + \angle OCB = 180°$ Angle sum in a triangle.

$\alpha + \alpha + \beta + \beta = 180°$

$2\alpha + 2\beta = 180°$

$\alpha + \beta = 90°$ as required



Question 2

RTP: For the given diagram, $\angle AEB = \angle ADB = \angle ACB$

Construction: OA, OB radii

Proof:

Let $\angle AOB = \alpha°$

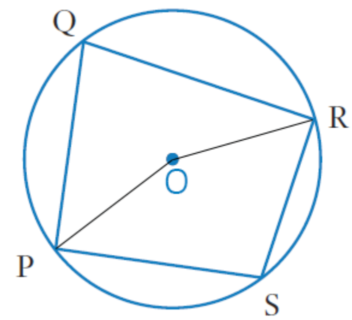
$\angle AEB = \frac{\alpha°}{2}$ Angle at the centre is twice the angle at the circumference

$\angle ADB = \frac{\alpha°}{2}$ Angle at the centre is twice the angle at the circumference

$\angle ACB = \frac{\alpha°}{2}$ Angle at the centre is twice the angle at the circumference

$\therefore \angle AEB = \angle ADB = \angle ACB = \frac{\alpha°}{2}$ as required

Question 3



RTP: For the given diagram, $\angle PQR + \angle PSR = 180°$

$\angle QPS + \angle QRS = 180°$

Construction: OP, OR radii

Proof:

Let $\angle POR = \alpha$

$\angle PQR = \frac{\alpha°}{2}$ Angle at the circumference is half the angle at the centre

Reflex $\angle POR = (360 - \alpha)°$ Angle sum around a point

$\angle PSR = \frac{(360-\alpha)°}{2}$ Angle at the circumference is half the angle at the centre

$= \left(180 - \frac{\alpha}{2}\right)°$

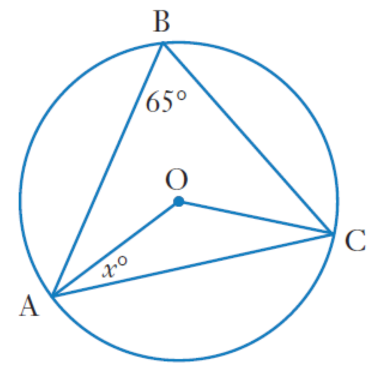
$\angle PQR + \angle PSR = \frac{\alpha°}{2} + 180° - \frac{a°}{2}$

$= 180°$ as required

Likewise, we can show $\angle QPS + \angle QRS = 180°$

Question 4

RTP: $x = 25$ for the diagram given diagram



Proof:

$\angle AOC = 130°$	Angle at the centre is twice the angle at the circumference
$OA = OC$	Radii of the same circle
$\angle OAC = \angle OCA = x°$	Base angles of an isosceles triangle
$\angle OAC + \angle OCA + \angle OAC = 180°$	Angle sum in a triangle.

$x + x + 130 = 180$

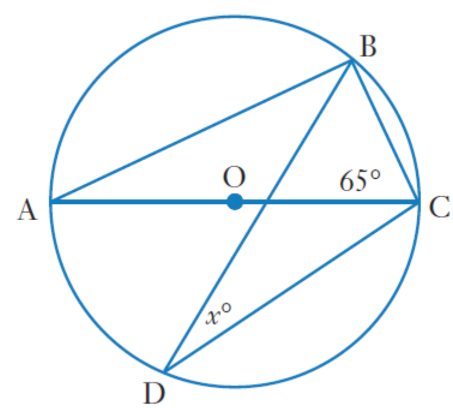
$2x + 130 = 180$

$2x = 50$

$x = 25$ as required

Question 5

RTP: Prove $x = 25$ for the given diagram



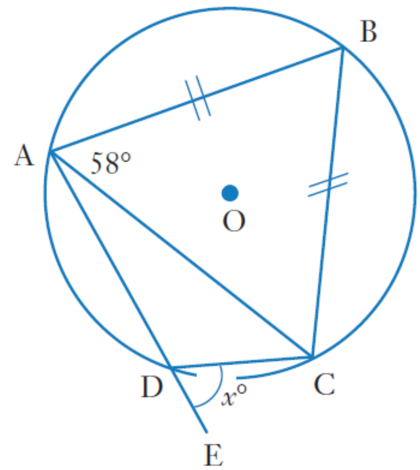
Proof:

AC is a diameter	Chord passing through centre
$\angle ABC = 90°$	Angle in a semicircle
$\angle BAC = 25°$	Angle sum of a triangle
$\angle BAC = \angle CDB$	Angles in the same segment are congruent

$x = 25$ as required

Question 6

RTP: $x = 64$ for the given diagram



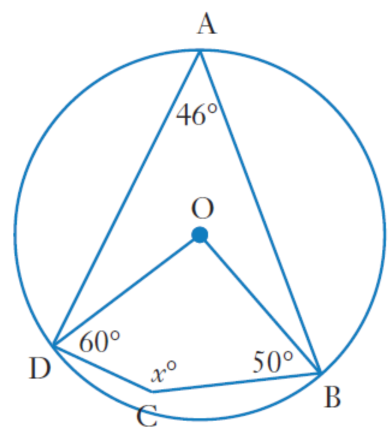
Proof:

$BA = BC$	Given
$\triangle ABC$ is isosceles	2 congruent sides
$\angle BAC = \angle BCA = 58°$	Base angles of an isosceles triangle
$\angle ABC = 180 - 116 = 64°$	Angle sum of a triangle
$\angle ADC = 116°$	Opposite angles of a cyclic quadrilateral are supplementary
$\angle CDE + \angle CDA = 180°$	Angles forming a straight line

$$x + 116 = 180$$
$$x = 64 \text{ as required}$$

Question 7

RTP: Prove $x = 158$ for the given diagram



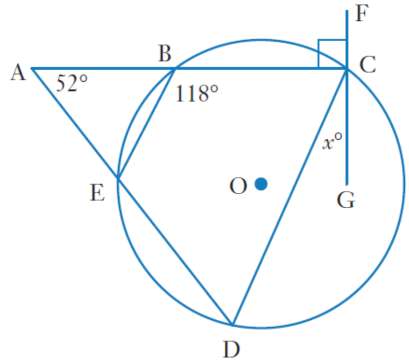
Proof:

$\angle DOB = 92°$	Angle at the centre is twice the angle at the circumference
$\angle ODC + \angle DCB + \angle CBO + \angle BOD = 360°$	Angle sum of a quadrilateral

$$\angle DCB = 360 - 60 - 50 - 92$$
$$= 158 \text{ as required}$$

Question 8

RTP: $x = 24$ for the given diagram



Proof:

$\angle ABE = 62°$	Angles forming a straight line are supplementary
$\angle AEB = 66°$	Angle sum of a triangle
$\angle BED = 114°$	Angles forming a straight line are supplementary
$\angle BCD = 66°$	Opposite angles of a cyclic quadrilateral are supplementary

Angles forming a straight line are supplementary

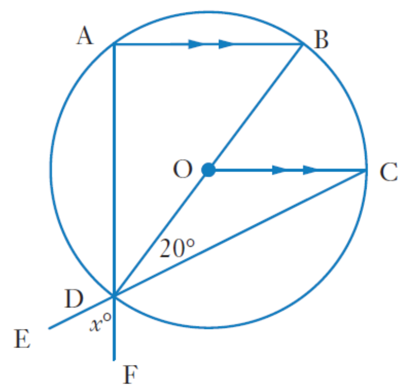
$\angle FCB + \angle BCD + \angle DCG = 180°$ Angles forming a straight line

$90 + 66 + x = 180$

$x = 24$ as required

Question 9

RTP: $x = 70$ for the given diagram

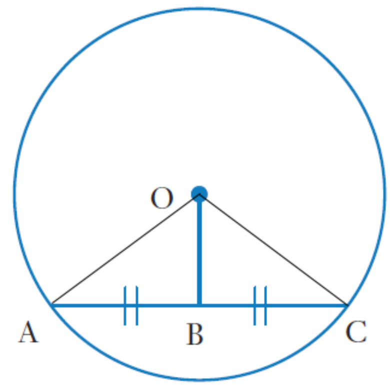


Proof:

$OC = OD$	Radii of same circle
$\angle OCD = 20°$	Base angles of isosceles triangle
$\angle COD = 140°$	Angle sum of a triangle
$\angle COB = 40°$	Angles forming a straight line are supplementary
$\angle OBA = 40°$	Alternate angle to $\angle COB$
$\angle BAO = 90°$	Angle in a semicircle
$\angle ADB = 50°$	Angle sum of a triangle
$\angle ADC = 70°$, $\angle EDF = 70°$ as required	Vertically opposite angle to $\angle ADC$

Question 10

RTP: $\angle OBA = \angle OBC = 90°$ in the given diagram



Construction: OA, OC radii

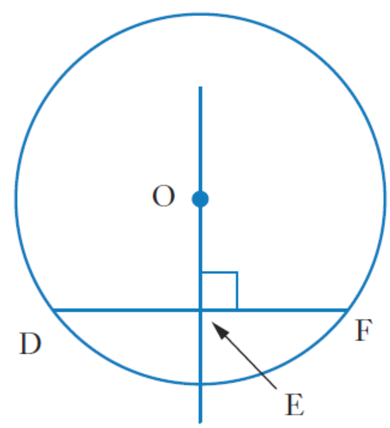
Proof:

$OA = OC$	Radii of same circle
$AB = CB$	Given OB Common
$\Delta OBA \cong \Delta OBC$	SSS congruence
$\angle OBA = \angle OBC$	Corresponding angles in congruent triangles
$\angle OBA + \angle OBC = 180°$	Angles forming a straight line are supplementary

$$2 \times \angle OBA = 180°$$
$$\angle OBA = 90°$$
$$\angle OBC = 90° \text{ as required}$$

Question 11

RTP: DE = EF in the given diagram

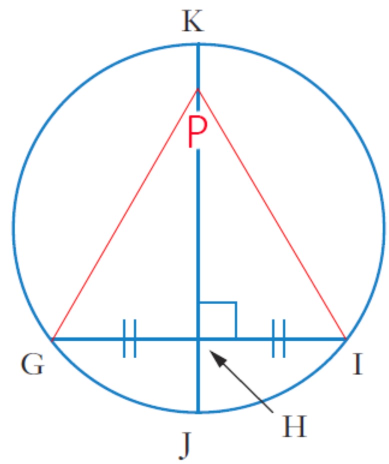


Construction: OD, OF radii

Proof:

$\angle OED = \angle OEF = 90°$	Angles on a straight line
$OD = OF$	Radii of the same circle
OE	Common
$\Delta OED \cong \Delta OEF$	RHS congruence
$\therefore DE = FE$ as required	Corresponding sides in congruent triangles

Question 12



RTP: JK passes through the centre of the circle for the given diagram

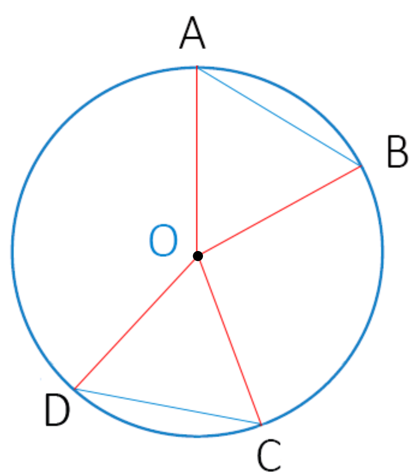
Construction: Let P be a point on JK

Proof:

$GI = HI$	Given
$\angle PIG = \angle PIH$	Given
PI	Common
$\Delta PGI \cong \Delta PHI$	RHS congruence
$PI = PG$	Corresponding sides in congruent triangles

For some particular P, PG and PI must equal the radius of the circle and therefore P must be the centre of the circle. As P is on JK, JK must pass through the centre of the circle. (as required)

Question 13



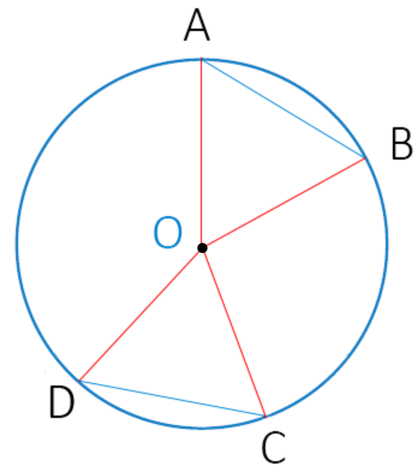
RTP: $\angle AOB = \angle COD$ in the given diagram

Constructions: OA, OB, OC and OD, radii

Proof:

$OA = OC$	Radii of same circle
$OB = OD$	Radii of same circle
$AB = CD$	Given
$\Delta OAB \cong \Delta OCD$	SSS congruency
$\angle AOB = \angle COD$ as required	Corresponding angles in congruent triangles

Question 14



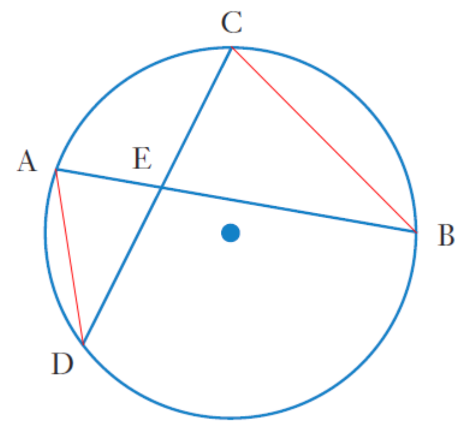
RTP: $AB = CD$ in the given diagram

Constructions: OA, OB, OC and OD, radii

Proof:

$OA = OC$	Radii of same circle
$OB = OD$	Radii of same circle
$\angle AOB = \angle COD$	Given
$\triangle OAB \cong \triangle OCD$	SAS congruency
$AB = CD$ as required	Corresponding sides in congruent triangles

Question 15



RTP: $AE \times EB = DE \times EC$ for the given diagram

Constructions: AD, CB chords

Proof:

$\angle DAB = \angle DCB$	Angles in the same segment
$\angle ADC = \angle ABC$	Angles in the same segment
$\angle AED = \angle CEB$	Vertically opposite angles
$\Delta AED \sim \Delta CEB$	AA similarity
$\frac{DE}{BE} = \frac{EA}{EC}$	Corresponding sides in similar triangles are in the same ratio

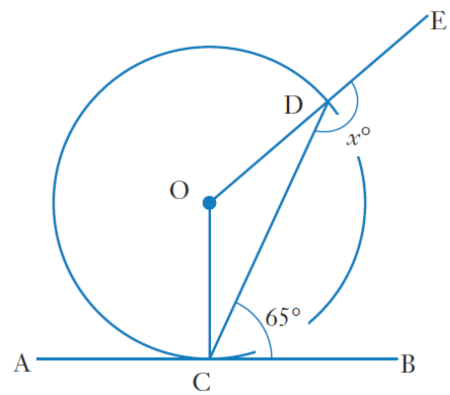
$\therefore AE \times EB = DE \times EC$ as required

From the above result:

$$AB \times BC = DB \times BE$$
$$4 \times 6 = x \times 8$$
$$24 = 8x$$
$$x = 3$$

$$ST \times TQ = PT \times TR$$
$$5 \times 6 = 10 \times TR$$
$$30 = 10 \times TR$$
$$TR = 3$$
$$y = PR$$
$$= PT + TR$$
$$= 10 + 3$$
$$= 13$$

Question 1



RTP: $x = 155$ for the given diagram

Proof:

$\angle OCB = 90°$ Angle between tangent and radius

$\angle OCD = 90 - 65 = 25°$

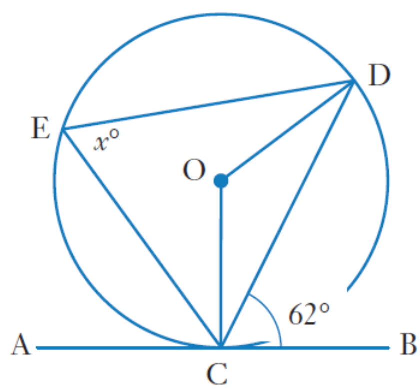
ΔOCD is isosceles OC, OD radii of same circle

$\angle ODC = \angle OCD = 25°$ Base angles of isosceles triangle

$\angle ODC + \angle EDC = 180°$ Angles forming a line

$\angle EDC = 180 - 25 = 155°$ as required

Question 2



RTP: $x = 62$ for the given diagram

Proof:

$\angle OCB = 90°$ Angle between tangent and radius

$\angle OCD = 90 - 62 = 28°$

ΔOCD is isosceles OC, OD radii of same circle

$\angle ODC = \angle OCD = 28°$ Base angles of isosceles triangle

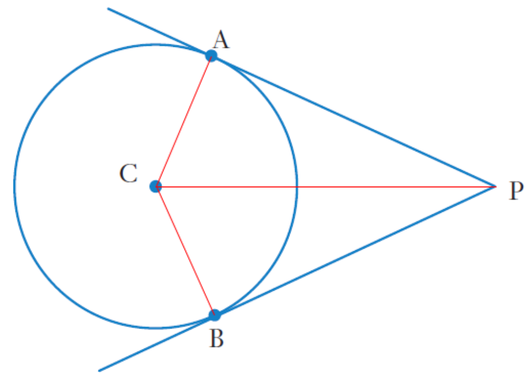
$\angle COD = 124°$ Angle sum of triangle

$\angle CED = 62°$ as required Angle at the centre is twice the angle at the circumference

Question 3

a RTP: PA = PB for the given diagram

Constructions: CA, CB radii, CP



Proof:

$\angle CAP = \angle CBP = 90°$	Angle between radius and tangent
$CA = CB$	Radii of same circle
CP	Common
$\triangle CAP \cong \triangle CBP$	RHS Congruency
$PA = PB$ as required	Corresponding sides in congruent triangles

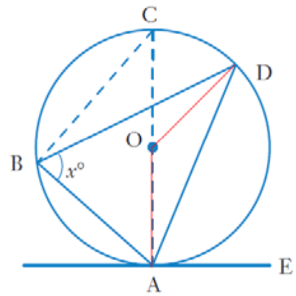
b RTP: PC bisects $\angle APB$

From part **a**,

$\Delta CAP \cong \Delta CBP$	RHS Congruency
$\angle CPA = \angle CPB$	Corresponding angles in congruent triangles

PC bisects $\angle APB$ as required.

Question 4



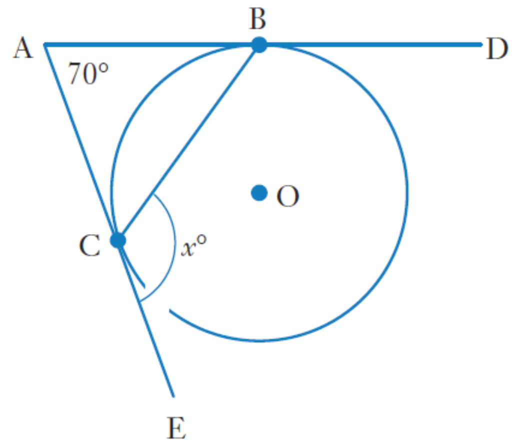
RTP: $\angle DBA \cong \angle DAE$ for the given diagram

Constructions: AO, OD radii

Proof:

$\angle AOD = 2x°$	Angle at the centre is twice the angle at the circumference
OA = OD	Radii of the same circle
ΔOAD is isosceles	
$\angle OAD = \angle ODA$	Base angles of isosceles triangle
$\angle OAD = \dfrac{180-2x}{2} = (90-x)°$	Angle sum of triangle
$\angle OAE = 90°$	Angle between radius and tangent
$\angle OAD + \angle DAE = \angle OAE$	
$\angle DAE = 90-(90-x) = x°$ as required	

Question 5



RTP: $x = 125$ in the given diagram

Proof:

$AB = AC$ Tangents from the same point are congruent

ΔABC is isosceles

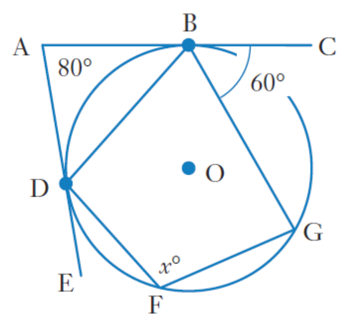
$\angle ACB = \angle ABC$ Base angles of isosceles triangle

$$\angle ACB = \frac{180 - 70}{2} = 55°$$ Angle sum of triangle

$\angle ACB + \angle BCE = 180°$ Angles forming a line

$\angle BCE = 180 - 55 = 125°$ as required

Question 6



RTP: $x = 110$ in the given diagram

Proof:

$AB = AD$ Tangents from the same point are congruent

ΔABD is isosceles

$\angle ADB = \angle ABD$ Base angles of isosceles triangle

$\angle ABD = \frac{180 - 80}{2} = 50°$ Angle sum of triangle

$\angle ABD + \angle DBG + \angle GBC = 180°$ Angles forming a line

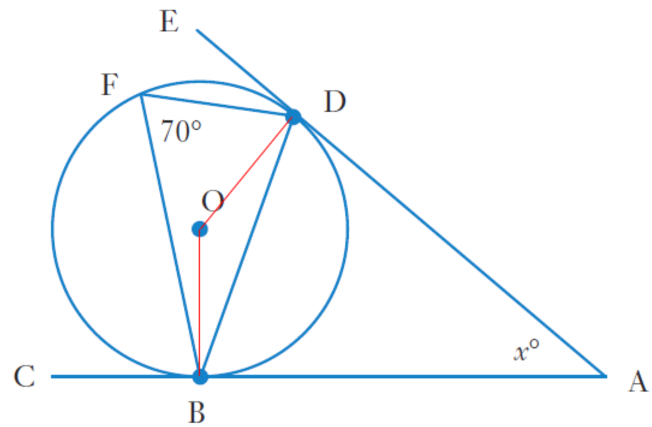
$\angle DBG = 180 - 50 - 60 = 70°$

$x = 110°$ as required Opposite angles in a cyclic quadrilateral

Question 7

RTP: $x = 40$ in the given diagram

Constructions: OD, OB radii

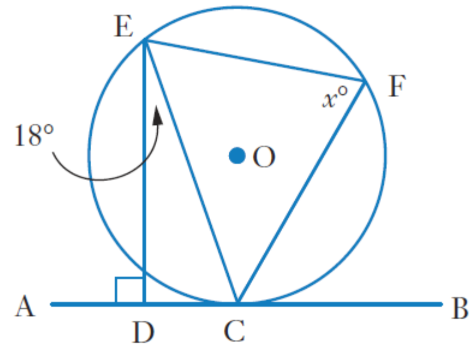


Proof:

$\angle BOD = 140°$	Angle at the centre is twice the angle at the circumference
ΔODB is isosceles	OB, OD radii of the same circle
$\angle ODB = \angle OBC$	Base angles of an isosceles triangle
$\angle OBD = \frac{180-140}{2} = 20°$	Angle sum of triangle
$\angle OBA = 90°$	Angle between radius and tangent
$\angle DBA = 90 - 20 = 70°$	
$AB = AD$	Tangents from the same point are congruent
ΔADB is isosceles	
$\angle BDA = \angle DBA = 70°$	Base angles of an isosceles triangle
$\angle BAD = 180 - (2 \times 70) = 40°$	Angle sum of triangle

Question 8

RTP: $x = 72$ for the given diagram

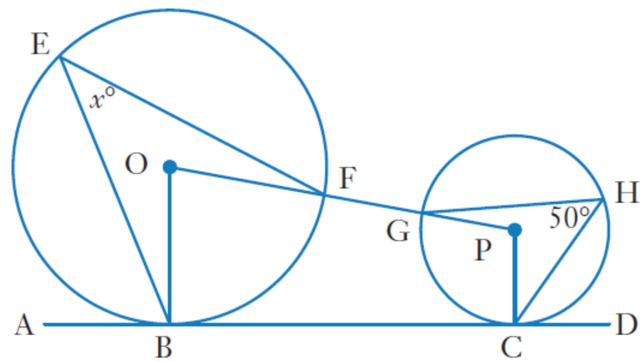


Proof:

$\angle CDE + \angle ADE = 180°$	Angles forming a line
$\angle CDE = 90°$	
$\angle DCE = 72°$	Angle sum of triangle
$\angle CFE = 72°$	Angle in alternate segment

Question 9

RTP: $x = 40$ for the given diagram

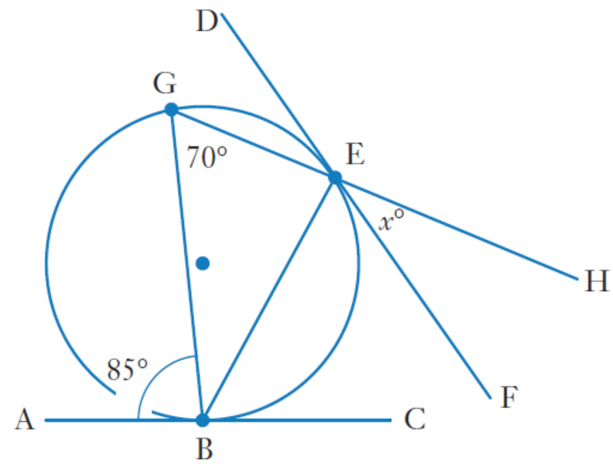


Proof:

$\angle CPG = 108°$	Angle at the centre is twice the angle at the circumference
$\angle BCP = 90°$	Angle between tangent and radius
$\angle CBO = 90°$	Angle between tangent and radius
$\angle BOF = 360 - (100 + 2 \times 90)$ $= 80°$	Angle sum of quadrilateral
$\angle BEF = 40°$ as required	Angle at the circumference is half the angle at the centre

Question 10

RTP: $x = 25$ in the given diagram



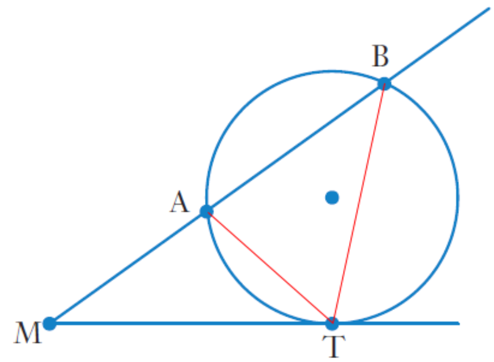
Proof:

$\angle BEG = 85°$	Angle in alternate segment
$\angle GBE = 25°$	Angle sum of triangle
$\angle DEG = 25°$	Angle in alternate segment
$\angle FEH = 25°$ as required	Vertically opposite angles

Question 11

RTP: $MT^2 = MA \times MB$ in the given diagram

Constructions: AT, BT

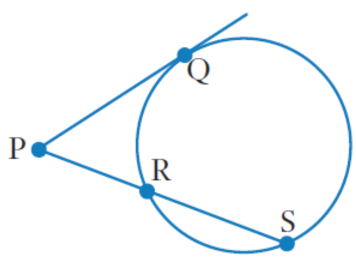


Proof :

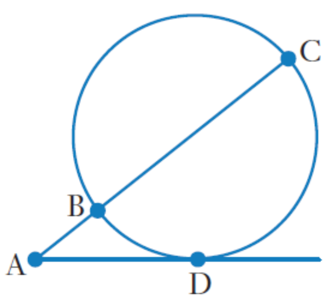
Let $\angle TMA = \alpha, \angle MTA = \beta$

$\angle TBA = \beta°$	Angle in alternate segment
$\angle TMA \sim \angle BMT$	AA similarity
$\frac{MT}{MB} = \frac{MA}{MT}$	Corresponding sides in similar triangles are in the same ratio

$MT^2 = MA \times MB$ as required



$$PQ^2 = PR \times PS$$
$$x^2 = 8 \times (8 + 10)$$
$$= 144$$
$$x = \pm 12 \text{ (disregard } x < 0)$$
$$x = 12 \text{ cm}$$

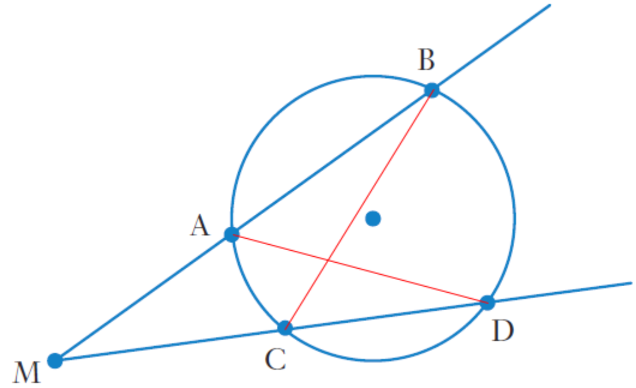


$$AD^2 = AB \times AC$$
$$20^2 = y \times (y + 30)$$
$$400 = y^2 + 30y$$
$$0 = y^2 + 30y - 400$$
$$= (y + 40)(y - 10)$$

$$y = -40, 10 \text{ (disregard } x < 0)$$
$$y = 10 \text{ cm}$$

Question 12

RTP: $MA \times MB = MC \times MD$ in the given diagram



Constructions: AD, BC

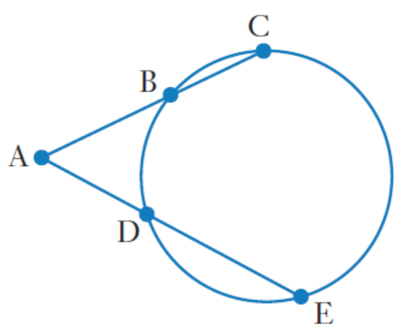
Proof :

In ΔMCB and ΔMAD

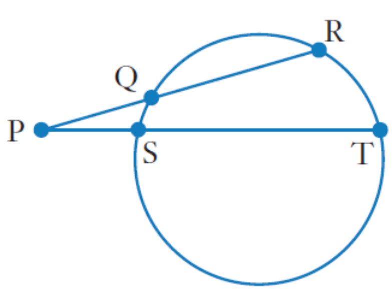
$\angle MBC = \angle MDA$	Angles in same segment
$\angle AMC$	Common
$\Delta MCB \sim \Delta MAD$	AA similarity

$\frac{MB}{MD} = \frac{MC}{MA}$ Corresponding sides in similar triangles are in the same ratio

$MA \times MB = MC \times MD$ as required



$$\begin{aligned} AB \times AC &= AD \times AE \\ 40 \times (40+60) &= 50 \times (50+x) \\ 50x + 2500 &= 4000 \\ 50x &= 1500 \\ x &= 30 \text{ cm} \end{aligned}$$



$$\begin{aligned} PQ \times PR &= PS \times PT \\ 6 \times 15 &= y \times (y+13) \\ 90 &= y^2 + 13y \\ 0 &= y^2 + 13y - 90 \\ &= (y+18)(y-5) \\ y &= -18, 5 \text{ (disregard } y < 0) \\ y &= 5 \text{ cm} \end{aligned}$$

Question 1

$^{13}C_8 = 1287$

$8! = 40\ 320$

Question 2

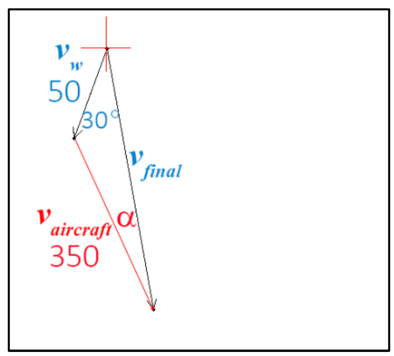
$^{3}C_2 \times {}^{6}C_4 = 45$

Question 3

$^{2}C_2 \times {}^{8}C_4 = 70$

$^{3}C_1 \times {}^{5}C_3 = 30$

Question 4



$$\frac{\sin 30°}{350} = \frac{\sin \alpha}{50}$$

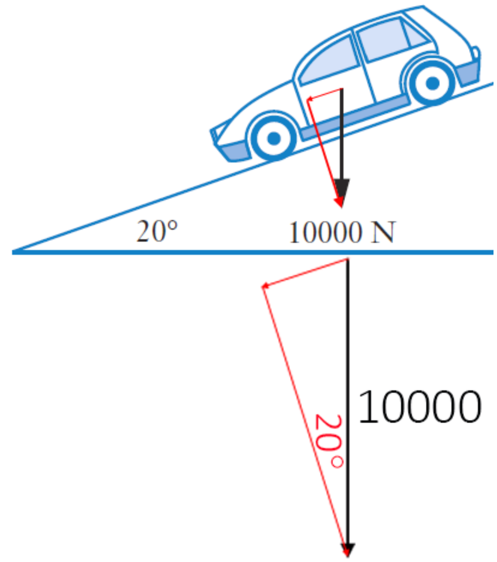
$$\alpha = 4°$$

Missing angle in triangle: 166°

Bearing of aircraft 166°.

Question 5

a



$$\sin 20° = \frac{|\mathbf{i}|}{10000}$$

$$|\mathbf{i}| = 10000 \times \sin 20°$$

$$= 3420$$

$$\cos 20° = \frac{|\mathbf{j}|}{10000}$$

$$|\mathbf{j}| = 10000 \times \cos 20°$$

$$= 9397$$

The 10 000N downward force is $(-3400\mathbf{i} - 9400\mathbf{j})$ N

b The horizontal force, $\mathbf{i}$ is being resisted by the brakes, $\therefore$ 3400 N

Question 6

6 out of 10 if I remove one mark per error.

Error 1: AD = DC should read AD = CD to match corresponding vertices

Error 2: The line stating congruence has named vertices in the incorrect order.

It should read Thus $\triangle ABD$ is congruent to $\triangle CBD$.

Error 3: The reason for congruence is incorrect, it should be SAS.

Error 4: The link between congruence and isosceles is not clearly established.

A statement similar to AB = CB (Corresponding sides in congruent triangles) needs to be added.

Question 7

a $\overrightarrow{\mathrm{AC}} = 2\mathbf{b}$

b $\overrightarrow{\mathrm{AF}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BF}}$

$= \mathbf{b} + \frac{1}{3}\mathbf{b}$

$= \frac{4}{3}\mathbf{b}$

c $\overrightarrow{\mathrm{DC}} = \overrightarrow{\mathrm{DB}} + \overrightarrow{\mathrm{BC}}$

$= \mathbf{a} + \mathbf{b}$

d $\overrightarrow{\mathrm{EC}} = \overrightarrow{\mathrm{ED}} + \overrightarrow{\mathrm{DC}}$

$= \mathbf{b} + \mathbf{a} + \mathbf{b}$

$= \mathbf{a} + 2\mathbf{b}$

e $\overrightarrow{\mathrm{EG}} = \overrightarrow{\mathrm{ED}} + \frac{1}{2}\overrightarrow{\mathrm{DC}}$

$= \mathbf{b} + \frac{1}{2}(\mathbf{a} + \mathbf{b})$

$= \frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$

f

$$\overrightarrow{GF} = \overrightarrow{GC} + \overrightarrow{CF}$$
$$= \frac{1}{2}\overrightarrow{DC} + \frac{2}{3}\overrightarrow{CB}$$
$$= \frac{1}{2}(\mathbf{a}+\mathbf{b}) + \frac{2}{3}(-\mathbf{b})$$
$$= \frac{1}{2}\mathbf{a} - \frac{1}{6}\mathbf{b}$$

$$\overrightarrow{GH} + \overrightarrow{HC} = \overrightarrow{GC}$$
$$h\overrightarrow{GF} + k\overrightarrow{EC} = \frac{1}{2}\overrightarrow{DC}$$
$$h(\frac{1}{2}\mathbf{a} - \frac{1}{6}\mathbf{b}) + k(\mathbf{a}+2\mathbf{b}) = \frac{1}{2}(\mathbf{a}+\mathbf{b})$$
$$\frac{h}{2}\mathbf{a} - \frac{h}{6}\mathbf{b} + k\mathbf{a} + 2k\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$
$$\frac{h}{2}\mathbf{a} + k\mathbf{a} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{b} + \frac{h}{6}\mathbf{b} - 2k\mathbf{b}$$
$$\left(\frac{h}{2} + k - \frac{1}{2}\right)\mathbf{a} = \left(\frac{1}{2} + \frac{h}{6} - 2k\right)\mathbf{b}$$

As $\mathbf{a}$ and $\mathbf{b}$ are not parallel vectors the only way for this statement to be true is if the coefficients of $\mathbf{a}$ and $\mathbf{b}$ equal zero.

$$\frac{h}{2} + k - \frac{1}{2} = 0 \Rightarrow k = \frac{1}{2} - \frac{h}{2}$$
$$\frac{1}{2} + \frac{h}{6} - 2k = 0$$
$$\frac{1}{2} + \frac{h}{6} - 2\left(\frac{1}{2} - \frac{h}{2}\right) = 0$$
$$\frac{1}{2} + \frac{h}{6} - 1 + h = 0$$
$$\frac{7h}{6} = \frac{1}{2}$$
$$7h = 3$$
$$h = \frac{3}{7}$$
$$k = \frac{1}{2} - \frac{h}{2} = \frac{1}{2} - \frac{3}{14} = \frac{2}{7}$$